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Round Table

"Open Problems"

Mathematical aspects of surface water waves

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1. The initial value problem

Problem 1. *It seems as though it is a natural question to show that, at least for sufficiently small initial data, solutions exist globally in time. That is, the flow* $\Phi_t(\eta, \xi)$ *should be shown to exist for all* $t \in \mathbb{R}$ *on a ball* $B_r(0)$ *of possibly small radius* r *about zero in an appropriate function space.*

Problem 2. *Use the coordinates in which the water waves problem has a Hamiltonian structure, in an essential way for the initial value problem.*

Problem 3. *How do solutions break down?*

2. The long wave and modulational limits

Problem 4. *Justify the* $d = 3$ *dimensional water wave models, as limits of the equations of free surface water waves.*

3. Traveling waves

Problem 5. *Elucidate the nature of the crescent wave patterns of doubly periodic traveling waves.*

Problem 6. *What is the nature of the singularity of crests of extremal traveling waves for the three-dimensional problem?.*

Problem 7. *Do truly three dimensional solitary waves exist? These cannot be everywhere non-negative without being the zero solution.*

Problem 8. *Give a Bloch theory for the stability of doubly periodic water wave patterns.*

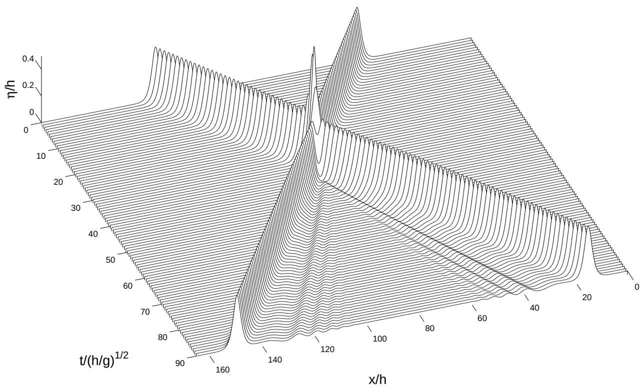

Figure 1. Head-on collision of two solitary waves of equal height $S/h = 0.4$, showing the run-up, the phase lag and the dispersive tail generated by the interaction. The amplitude of the post-collision solitary waves is slightly less than before the collision, being measured as $S^+/h = 0.3976$ at time $t/\sqrt{h/g} = 90$. The post-collision solitary waves are also delayed slightly from their pre-collision asymptotically linear trajectories. This phase lag is measured to be $(a_j - a_j^+)/h = 0.3257$, where a_j (respectively a_j^+) are the t-axis intercepts of the asymptotically linear trajectories before (respectively after) the collision.

4. Invariant structures in phase space

Problem 9. *Prove that there exist (Cantor) parameter families of quasi-perodic and/or almost periodic solutions for the water waves problem.*

Problem 10. *Prove that a soliton - scattering picture occurs, roughly as illustrated in Figure 1. That is, general initial data, suitable localized, resolves itself under time evolution into two trains of solitary waves plus a residual, and the latter is described by a scattering operator related to the linear problem.*

5. Wave turbulence

Problem 11. *Elucidate this approach as a kinetic theory of wave field propagation and interaction. Provide, to the extent possible, a mathematically rigorous derivation of the kinetic equations of wave turbulence.*

Problem 12. *Prove that power law Kolmogorov – Zakharov spectra exist, and that they govern the large time asymptotics of wave fields, at least in certain situations. Find the full set of solutions of $S_{nl} = 0$, and their respective macroscopic variables, and derive the appropriate macroscopic equations which determine their evolution.*

Statistical hydrodynamics and mathematics of turbulence

Sergei Kuksin

Consider the randomly forced Navier-Stokes equation in dimension $d = 2$ or $d = 3$:

$$\dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \eta(t,x), \quad \operatorname{div} u = 0. \tag{1}$$

We restrict ourselves to the case when $x \in T^d$ or $x \in \mathbb{R}^d$. If $x \in \mathbb{T}^d$, we assume that $\int \eta\, dx \equiv \int u\, dx \equiv 0$. The force $\eta(t,x)$ is a random field which

i) as a function of x is smooth, homogeneous (=stationary) and non-degenerate,

ii) as a function of t is a white noise.

See [Kuk06, Kup].

I) $x \in \mathbb{T}^2$. In this case (1) defines a Markov process in the space H of smooth divergence-free vector-fields with zero meanvalue. This process has a unique stationary measure μ_ν and the distribution of any solution converges to this measure exponentially fast. Every solution, viewed as a random process in H, satisfies the Strong Law of Large Numbers and CLT. See [Kuk06] for discussions and references.

Let $u_\nu(t,x)$ be a stationary in time solution of (1). Then $\mathcal{D}(u_\nu(t,\cdot)) = \mu_\nu$. Here $u_\nu(t,\cdot)$ is the solution, interpreted as a random process in the function space H, and $\mathcal{D}(u_\nu(t,\cdot))$ is its distribution for a fixed time t (i.e., a measure in H). The main problem of turbulence for a $2d$ space-periodic flow is the following:

Problem 1^***.** Study the correlation

$$\mathbf{E} u^i(t,x) u^j(t,x+\xi), \quad i,j = 1,2\,,\ x \in \mathbb{T}^2, \tag{2}$$

when $|\xi| \ll 1$ and the viscosity ν is small.

This is a notoriously difficult question. In particular, since

– behaviour of the function (2) for $|\xi| \ll 1$ is believed to depend on the relation between $|\xi|$ and ν;

– it is unclear how to split this problem to simpler questions;

– 'right' model problems are unknown.

On a formal level Problem 1 admits a reformulation in spectral terms, which is equally popular:

Problem 2^***.** Let $\{\hat{u}_{\mathbf{k}}(t)\}$ be Fourier coefficients of $u(t,x)$. Define $E_{\mathbf{k}} = \mathbf{E}|\hat{u}_{\mathbf{k}}(t)|^2$. The task is to study $E_{\mathbf{k}}$ when $|\mathbf{k}| \gg 1$.

Often instead of the energy $E_{\mathbf{k}}$ of a wave-vector $\mathbf{k}$ people study the energy of a wave-number k, loosely defined as $(2L)^{-1}\sum_{\left||\mathbf{k}|-k\right|\le L} E_{\mathbf{k}}^2$. The width L of the averaging layer is ≥ 1. Its 'right' value may depend on ν, but should not be 'too big'.

It is believed that the limiting properties in Problems 1 and 2 are universal in the sense that they are independent from detailed properties of the force η. It is known that, indeed, the measures μ_ν possess some universal properties: they satisfy infinitely many algebraical relations, depending only on one scalar characteristic of η. See [Kuk06], Section 11.

It is natural to wish to scale the force in (1) in such a way that the energy of the stationary solution $E^\nu = \int |u_\nu(t,x)|^2\,dx$ remains of order one as $\nu \to 0$. To achieve this goal one has to replace the force η by the scaled force

$$\eta = \sqrt{\nu}\,\eta_0(t,x), \tag{3}$$

where η_0 is a fixed random force of order one, satisfying i), ii). If we do so, then along a subsequence $\nu_j \to 0$ the measure μ_ν converges to a limiting measure μ_0 (which a priori depends on the sequence). The measure μ_0 is an invariant measure for the Euler equation

$$\dot u + (u\cdot\nabla)u + \nabla p = \eta(t,x), \quad \operatorname{div} u = 0\,,$$

interpreted as a dynamical system in suitable function space of vector-fields. See [Kuk06], Section 10. But the Euler equation has infinitely many integrals of motions, so it has a lot of different invariant measures. How to distinguish among them the limiting measure μ_0?

Problem 3. Study properties of a limiting measure μ_0. Does the measure (or its 'relevant properties') depend on the sequence $\{\nu_j\}$? Study asymptotics of the correlation tensor $\int u^i(x)u^j(x+\xi)\,\mu_0(du)$ as $\xi \to 0$, and of the energies $\int |\hat u_{\mathbf{k}}|^2\,\mu_0(du)$ as $\mathbf{k} \to \infty$. Do they depend on the sequence $\{\nu_j\}$?

For the equation with the force, scaled as (3), the energy of the stationary solution E^ν has a finite ν-independent exponential moment (see [Kuk06]). Accordingly, $\mathbf{P}\{E^\nu \ge C\} \le ce^{-\sigma C}$ for any $C > 0$, with some positive σ and c.

Problem 4. Find a lower bound for $\mathbf{P}\{E^\nu \ge C\}$ when $C \to \infty$.

Similar, it is interesting to find lower and upper bounds for $\mathbf{P}\{E^\nu \le C\}$, where $C \ll 1$.

II. $x \in \mathbb{R}^2$. Here the class of solutions, the most relevant for the theory of turbulence, is formed by solutions, homogeneous in x (and not necessarily periodic). Let the force η be smooth homogeneous in x and white in t.

Problem 5. a) Under what assumption on η the problem (1) has a unique solution, stationary in x?

b) Under what assumption this solution remains bounded (in a suitable sense) as $t \to \infty$?

c) Under what assumption the equation has a unique stationary measure, homogeneous in x?

The Problems 1 and 2 admit natural reformulations for the unique stationary measure as in c).

III. $x \in \mathbb{T}^3$. It seems that in the $3d$ case the main problem is the existence and uniqueness of a solution:

Problem 6*. Let $x \in \mathbb{T}^3$ and η satisfies i), ii) with $x \in \mathbb{T}^3$. Prove or disprove that

a) for a.a. ω eq. (1) has a unique smooth solution;

b) eq. (1) has a unique martingale ("weak") solution.

If a solution for eq. (1) is unique, then it defines a Markov process in a suitable function space $\{u(x)\}$ and has an invariant measure. For this measure Problems 1 and 2 have a natural reformulation. See in [Kup].

Note that the questions a) and b) remain non-trivial even if the force η is small. We also note that interesting and informative results about the $3d$ equation (1) can be obtained without knowing that its solution is unique. See recent relevant works by Da Prato, Debussche, Flandoli and others.

References

[Kuk06] S. B. Kuksin, *Randomly forced nonlinear PDEs and statistical hydrodynamics in 2 space dimensions*, Europear Mathematical Society Publishing House, 2006, also see mp_arc 06-178.

[Kup] A. Kupiainen, *Statistical theories of turbulence*, Random Media 2000 (Madralin, June 2000), to appear.

E. Feireisl

Compressible, viscous, and heat conducting fluids: Recent progress and open problems

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The mathematical theory of compressible, viscous, and heat conducting fluids represents a challenging topic in the mathematical fluid mechanics. The time evolution of these fluids is classically described by the Navier-Stokes-Fourier system that represents a mathematical formulation of the three basic physical principals: the conservation of mass, momentum, and energy. The resulting systen of partial differential equations is of mixed type: the continuity equation is hyperbolic, the momentum and energy equations can be classified as parabolic although the leading time derivative contains the density as a multiplier.

• *A priori estimates:* Similarly to hyperbolic systems of conservation laws, the balance of the total energy gives rise to *a priori* estimates in the natural "energy" norm. That is to say the quantities like the density, the velocity, and the temperature are bounded in suitable Lebesgue spaces in terms of the initial data. As is well-known, those estimates are not sufficient to prevent oscillations of the solution family both in time and space. Fortunately, as the dissipative mechanism due to viscosity and thermal conductivity is present, more estimates are available on the spatial gradient of the temperature and velocity preventing thus their spatial oscillations. There is no bound of this type, however, for the density. The available estimates for the velocity are basically the same as for the incompressible Navier-Stokes system, in particular, they do not seem to be sufficient for obtaining classical solutions. Thus the natural open problem is the same as for the incompressible Navier-Stokes system: Do classical solutions exist for all time? The second and completely open question is, of course, uniqueness of the weak solutions.

• *Estimates on the density:* The crucial quantity in the analysis of a compressible fluid is the density. The major open problem reads: Does the density stay bounded, both from above and away from zero, provided its initial distribution enjoys this property? In particular, it is not known whether the weak solutions can develop vacuum regions or concentrations in a finite time. The following properties seem to be "equivalent": **(i)** the density is bounded below away from zero for any positive time, **(ii)** the divergence is uniformly bounded, **(iii)** the density is bounded from above. There seem to be a piece of numerical evidence of vacui in the weak solutions for compressible Navier-Stokes equations but the final answer is not known.

• *The energy transfer:* Another challenging problem is the transfer of energy between its kinetic and caloric (internal) component. The entropy production for the weak solutions, in general, may contain a measure with a non-trivial singular part. The problem becomes particularly delicate when the vacuum is present. Let us note in passing that the original model was not supposed to describe the vacuum states, for which it does not yield the physically relevant solution.

Control problems in hydrodynamics

A. Shirikyan

Controllability of Navier–Stokes and Euler equations by a body force

Let us consider the Navier–Stokes system

$$\dot{u} + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \quad x \in D. \tag{1}$$

Here $D \subset \mathbb{R}^d$ is a bounded domain and $f = h + \eta$, where h is a given function and η is a control with range in a finite-dimensional space $E \subset C^\infty(D, \mathbb{R}^d)$. It was established by Agrachev and Sarychev that, in the case $d = 2$ and $D = \mathbb{T}^2$, the Navier–Stokes system (1) is approximately controllable and exactly controllable in finite-dimensional projections. These results were extended later by Rodrigues to case of square with Lions boundary condition and by Shirikyan to the 3D case with periodic boundary conditions. Similar properties hold for the Euler equations (both in 2D and 3D).

The following questions remain completely open:

- Exact controllability of (1) by a finite-dimensional force for analytic data.
- Controllability properties of (1) for other domains, e.g., a ball or a square with Dirichlet boundary condition.

Boundary control for Navier–Stokes and Euler equations

From the point of view of applications, the boundary control is more natural than the control by body force considered in the previous section. It was proved in a series of papers by Coron, Fursikov, Immanuilov that the Navier–Stokes system (1) is exactly controllable by a control supported by the boundary of D. However, their result does not give much information about the control. In particular, the following natural question arises:

- Controllability of (1) by means of a finite-dimensional boundary control.

Feedback stabilization of Navier–Stokes equations

Consider Eqs. (1) in which

$$f(t,x) = h(t,x) + (Ku)(t,x), \tag{2}$$

where h is a given function and K is a (nonlinear) operator defined on the phase space H and taking values in a finite-dimensional space E. The feedback stabilization problem for (1) can be formulated as follows: *given a solution* $\hat{u}$ *of* (1) *with* $f = h$*, construct an operator* $K_{\hat{u}} : H \to E$ *such that for any initial function* $u_0 \in H$ *we have*

$$\|u(t) - \hat{u}(t)\| \to 0 \quad \textit{as } t \to \infty,$$

where u *denotes the solution of* (1), (2) *with* $K = K_{\hat{u}}$ *that satisfies the initial condition* $u(0) = u_0$*.*

In the case when h does not depend on time and $\hat{u}$ is a stationary solution, the stabilization problem was studied by Fursikov, Barbu, Triggiani. They established feedback stabilization by means of finite-dimensional controllers supported either by the boundary or by a given subdomain of D. However, the control space E depends on $\hat{u}$. The following question is important in the ergodic theory of Navier–Stokes equations perturbed by a random force:

- Construct a finite-dimensional subspace $E \subset C^\infty(D, \mathbb{R}^d)$ and a stabilizer $K_{\hat{u}} : H \to E$ for (1).

Controllability for the Schrödinger equation with cubic nonlinearity

Controllability results discussed in the first section uses the quadratic structure of the nonlinear term of the Navier–Stokes system. In this regard, it would be interesting to study the following problem:

- Controllability of the nonlinear Schrödinger equation

$$\dot{u} - i\Delta u \pm i|u|^2 u = h(x) + \eta(t, x), \tag{3}$$

  where η is control function with range in a finite-dimensional space.

Problems on the Euler equations

A. Shnirelman

(I.) In 2-d case the E.e. have global in time regular solution, while in 3-d this is still unknown. Hence, the most interesting problems in 2-d pertain to the long time behavior of solutions, while in 3-d the problems revolve around the Singularity Problem and what happens if the singularity really exists. Here I consider only 2-d problems.

(II) Let M be a bounded domain in $\mathbb{R}^2$, or a compact Riemannian 2-d manifold with(out) a smooth boundary ∂M. Consider the smooth solutions $u(x,t)$ of E.e. in M, satisfying the boundary condition $u_n \big|_{\partial M} = 0$.

(1) What happens with a typical solution $u(x,t)$ of E.e. (with no external forces) as $t \to \infty$? Experiments and numerical simulations show that typically solution develops large scale structures, even if started from small-scale initial velocity. Moreover, it appears that a typical solution always has a limit as $t \to \infty$, which is a stationary and stable solution of E.e. Thus, the ideal incompressible fluid behaves irreversibly, while its laws of motion are time-reversible. This paradox is similar to the irreversible behavior of the systems studied by statistical mechanics (like gases of elastic particles).

(2) Let V be the vector space of incompressible vector fields in M. The E.e. define a dynamical system in V, whose flow we denote by $S_t : V \to V$. The above conjecture means that there exists an attracting set $\mathfrak{M} \subset V$, namely the set of steady and stable flows (attracting means attracting in the L^2 (energy) norm in V). It is known that $\mathfrak{M}$ is an infinite-dimensional precompact set in V (precompact in the L^2-sense), but little more. It is unknown, how many components it has; what is its Kolmogorov ε-entropy; whether it is homogeneous, or different pieces of $\mathfrak{M}$ have different ε-entropy.

(3) The irreversible character of dynamics of 2-d fluid can be justified by construction of Liapunov function(s), i.e. function $L(v), v \in V$, globally defined in V and continuous, s.t. $L(v)$ grows along any trajectory of the dynamical system (V, S_t):

$$\frac{d}{dt} L(S_t v) \geq 0,$$ and is > 0 "almost everywhere" in V (say, everywhere outside some set of codimension ∞).

The Liapunov functions like these really exist, but they describe the growth of weak singularities of solution, and have little physical meaning. Do there exist more "material" Liapunov functions?

(4) The Euler equations have well known energy and vorticity integrals, as well as momentum integral for domains with a symmetry (like the angular momentum in the disk). Do there exist other integrals?

~~Both new integrals and~~

Discovery of a new integral, as well as demonstration of its nonexistence, would be very interesting.

(5) For a smooth solution $u(x,t)$ the norm of its derivatives $\partial_x^\alpha u(x,t)$ usually grow as $t\to\infty$. V. I. Yudovich called this phenomenon "Deterioration of regularity". There exist some upper estimates of the norm of solution in C^k (or other space), possibly exaggerated

However, there are no estimates from below. This problem is important, because the norm of solution in C^k is a candidate for a Liapunov function.

The question can be asked also in the Lagrangian setup. ~~Let~~ Denote by $\mathcal{D}(M)$ the group of area-preserving diffeomorphisms of M. Let $\xi_t \subset \mathcal{D}(M)$, ~~$0 \le t \le$~~ $t \in \mathbb{R}$, be the flow defined by the solution $u(x,t)$ of E.e. Define complexity of ξ_t, $\mathcal{C}(\xi_t)$, in the following way. Let U_N be a small C^N-neighborhood of the identity map Id in $\mathcal{D}(M)$. For any $\xi \in \mathcal{D}$, $\mathcal{C}(\xi)$ is defined as the minimal number of maps $\eta_1, \dots, \eta_k \in U_N$

s.t. $\xi = \eta_1 \circ \dots \circ \eta_k$. What can be said on the asymptotic behavior of $\mathcal{C}(\xi_t)$ as $t \to \infty$? I suppose that $\log \mathcal{C}(\xi_t)$ is not sensitive to N and the size of U. Can we give a realistic estimate of $\mathcal{C}(\xi_t)$ from below?

⑥ There is a wide gap between the Arnold stable stationary flows and exponentially unstable ones. For example, consider the parallel flows in a periodic strip having the velocity profile $U(y)$, $0 \le y \le 1$. If, for example, $U(y) = y + \varepsilon \sin 20y$, then for ε small enough, the linearized problem has no unstable eigenvalues, hence there is no rough, exponential instability. However, this flow is not Arnold stable. So, is it stable in the space H^1 (vorticity space), or it is not?

I suppose it is unstable, but the growing perturbation is ~~such~~ so delicate and requires such a fine tuning that "the instability is very unstable itself". So, practically this flow is stable.